

solving a recurrence relation, we mean finding a closed form.
 we look at two basic methods: • Forward Substitution • Backward Substitution.

Example 2.4.7. Find a solution for the recurrence relation given by $a_n = a_{n-1} + 2n + 1$ with $a_0 = 4$.

Method 1 (forward Substitution)

$$a_0 = 4$$

$$a_1 = a_0 + 2 \times 1 + 1 = 4 + 2 \times 1 + 1$$

$$a_2 = a_1 + 2 \times 2 + 1 = 4 + 2 \times 1 + 1 + 2 \times 2 + 1$$

$$a_3 = a_2 + 2 \times 3 + 1 = 4 + 2 \times 1 + 1 + 2 \times 2 + 1 + 2 \times 3 + 1$$

see a pattern?

$$a_n = 4 + (2 \times 1 + 2 \times 2 + \dots + 2 \times n) + (\overbrace{1+1+\dots+1}^{n \text{ times}})$$

$$\stackrel{\text{simplify}}{=} 4 + 2 \cdot (1 + 2 + \dots + n) + n$$

$$= 4 + 2 \cdot \frac{n(n+1)}{2} + n$$

$$= 4 + n^2 + n + n$$

$$\therefore \boxed{a_n = n^2 + 2n + 4}$$

Method 2 (Backward Substitution)

$$a_n = a_{n-1} + 2n + 1$$

$$= a_{n-2} + 2(n-1) + 1 + 2n + 1$$

$$= a_{n-3} + 2(n-2) + 1 + 2(n-1) + 1 + 2n + 1$$

$$= \dots \text{ (see a pattern) } ??$$

$$= a_1 + 2 \times 2 + 1 + 2 \times 3 + 1 + \dots + 2(n-1) + 1 + 2n + 1$$

$$= a_0 + 2 \times 1 + 1 + 2 \times 2 + 1 + \dots + 2n + 1$$

$$= 4 + 2(1 + 2 + \dots + n) + (\underbrace{1 + \dots + 1}_{n \text{ times}}) \stackrel{\text{as before}}{=} \dots = n^2 + 2n + 4$$

$$\begin{aligned}
 &= \frac{5(1 - \frac{1}{512})}{\frac{1}{2}} + \frac{3(1 - 512)}{-1} \\
 &= 10(1 - \frac{1}{512}) + (3)(511) \\
 &= 10 \cdot \frac{511}{512} + 3 \cdot 511 = 511 \left(\frac{10}{512} + 3 \right) = 511 \cdot \frac{1546}{512} = \frac{790006}{512}
 \end{aligned}$$

Use
your
calculator

≈ 1542.98

$$\sum_{j=0}^n ar^j = a \frac{(1-r^{n+1})}{1-r}$$

2.4. SEQUENCES AND SUMMATIONS

CHAPTER 2.

Example 2.4.10. Compute $\sum_{i=0}^{10} \left(\sum_{j=0}^8 i \cdot 2^j \right)$

← read from inside

$$\begin{aligned}
 &= \sum_{i=0}^{10} \left(\sum_{j=0}^8 i \cdot 2^j \right) \quad \leftarrow \text{this is the geometric summation with } a=i, r=2 \\
 &= \sum_{i=0}^{10} \left(i \frac{(1-2^9)}{1-2} \right) \quad \leftarrow 511 \\
 &= \sum_{i=0}^{10} 511 i \quad \leftarrow \text{factoring the constant } 511 \\
 &= 511 \left(\sum_{i=0}^{10} i \right) \\
 &= 511 (0 + 1 + 2 + 3 + \dots + 10) \quad \leftarrow \text{formula} \\
 &= 511 \left(\frac{10 \cdot (10+1)}{2} \right) = 511 \times 55
 \end{aligned}$$

TABLE 2 Some Useful Summation Formulae.

Sum	Closed Form
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Summation Notation: To express the summation $a_m + a_{m+1} + \dots + a_n$, we may use the more precise notation

$$\sum_{j=m}^n a_j \quad \text{or} \quad \sum_{m \leq i \leq n} a_i \quad \text{or} \quad \sum_{i \in \{m, m+1, \dots, n\}} a_i$$

Example 2.4.8. Express using the $\sum$ notation.

(a) $1 + r + r^2 + \dots + r^n = \sum_{i=0}^n r^i \quad \text{or} \quad \sum_{j=1}^{n+1} r^{j-1} \quad \text{or} \dots$

(b) $1 + \frac{1}{2} + \frac{1}{3} + \dots = \sum_{i=1}^{\infty} \frac{1}{i}$

(c) $2^2 + 3^2 + 5^2 + 7^2 = \sum_{k \in \{2, 3, 5, 7\}} k^2 \quad \text{or} \quad \sum_{k \in \{2, 3, 5, 7\}} k^2$

Theorem 2.4.1 (Geometric Series). Let $a, r \in \mathbb{R}$ with $r \neq 0, 1$. Then

$$(1+r)(1-r) = 1-r^2$$

$$(1+r+r^2)(1-r) = 1-r^3$$

$$a + ar + ar^2 + \dots + ar^n = \sum_{j=0}^n ar^j = \frac{a(1-r^{n+1})}{1-r}$$

Proof. Note that $(1+r+r^2+\dots+r^n)(1-r)$

$$= (1+r+r^2+\dots+r^n) - (r+r^2+\dots+r^{n+1})$$

$$= 1 - r^{n+1}$$

We have $a + ar + ar^2 + \dots + ar^n = a(1+r+r^2+\dots+r^n)$

$$= \frac{a(1+r+r^2+\dots+r^n)(1-r)}{1-r}$$

$$= \frac{a(1-r^{n+1})}{1-r} \quad \square$$

Example 2.4.9. Compute $\sum_{j=0}^8 5 \cdot (0.5)^j + 3 \cdot 2^j$

Geometric Summation formula

$$= \sum_{j=0}^8 5 \cdot (0.5)^j + \sum_{j=0}^8 3 \cdot 2^j$$

$$= \frac{5(1-0.5^9)}{1-0.5} + \frac{3(1-2^9)}{1-2}$$

$$= \frac{5(1-\frac{1}{512})}{\frac{1}{2}} + \frac{3(1-512)}{-1}$$

$$= 10(1-\frac{1}{512}) + (3)(511)$$

$$= 10 \cdot \frac{511}{512} + 3 \cdot 511 = 511 \left(\frac{10}{512} + 3 \right) = 511 \cdot \frac{1546}{512} = \frac{790006}{512}$$

Use your calculator

$$\approx 1542.98$$

2.4 Sequences and Summations

Sequences are ordered list of numbers. For example, $(5, 7, 9, 11, 13, \dots)$ is a sequence. Formally:

Definition 2.4.1 (Sequence, n -th term, Sequence representation).

- A sequence is a function of the form $a: S \rightarrow \mathbb{R}$ with $S \subseteq \mathbb{Z}$.

Typically, we choose $S = \mathbb{N} = \{1, 2, 3, \dots\}$ or $S = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \dots\}$

- The n^{th} term of the sequence $a: S \rightarrow \mathbb{R}$ is simply $a(n)$.

Typically, we write a_n for $a(n)$.

- If $a: \mathbb{N} \cup \{0\} \rightarrow \mathbb{R}$ is a sequence it's usually represented as $(a_n)_{n=0}^{\infty}$ or $(a_0, a_1, a_2, \dots)$ or (a_n)

Example 2.4.1. $a_n = \frac{1}{n}$.

Means the sequence $(\frac{1}{n})_{n=1}^{\infty}$ or $(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$

Definition 2.4.2 (Geometric Progression).

A sequence of the form $(a \cdot r^n)_{n=0}^{\infty} = (a, ar, ar^2, ar^3, \dots)$

where a and r are constants, is called a geometric progression.

Called: initial term

Called: common ratio

Example 2.4.2. Write the first 5 terms of a geometric progression with the given initial value a and common ratio r .

- | | | |
|----------------------|---|--|
| (a) $a = 10, r = -1$ | $(10, -10, 10, -10, 10, \dots)$ | $= (10 \cdot (-1)^n)_{n=0}^{\infty}$ |
| (b) $a = 2, r = 5$ | $(2, 10, 50, 250, 1250, \dots)$ | $= (2 \cdot 5^n)_{n=0}^{\infty}$ |
| (c) $a = 6, r = 1/3$ | $(6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \dots)$ | $= (6 \cdot (\frac{1}{3})^n)_{n=0}^{\infty}$ |

Definition 2.4.3 (Arithmetic Progression).

A sequence of the form $(a + dn)_{n=0}^{\infty} = (a, a+d, a+2d, a+3d, \dots)$

where a and d are constants is called an arithmetic progression

initial value

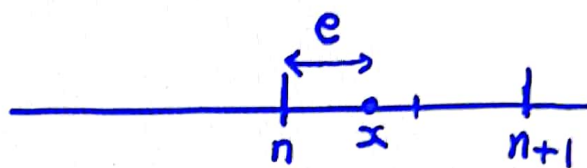
common difference

Example 2.3.9. Prove that if $x \in \mathbb{R}$ then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$.

Let $x = n + e$ where

$n \in \mathbb{Z}$ and $e \in \mathbb{R}$

$0 \leq e < 1$.



Consider two cases:

Case 1 $0 \leq e < \frac{1}{2}$

$$\text{Then } \lfloor 2x \rfloor = \lfloor 2(n+e) \rfloor = \lfloor 2n + 2e \rfloor = 2n$$

$$\lfloor x \rfloor = \lfloor n + e \rfloor = n$$

$$\lfloor x + \frac{1}{2} \rfloor = \lfloor n + e + \frac{1}{2} \rfloor = n$$

integer less than 1
↓

int. less than 1
↓

$$\text{Since } 2n = n + n \quad \therefore \lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$$

Case 2 $\frac{1}{2} \leq e < 1$

int.
↓

between 1 and 2
↓

$$\text{Then } \lfloor 2x \rfloor = \lfloor 2n + 2e \rfloor = 2n + 1$$

$$\lfloor x \rfloor = \lfloor n + e \rfloor = n$$

$$\lfloor x + \frac{1}{2} \rfloor = \lfloor n + e + \frac{1}{2} \rfloor = n + 1$$

int.
↓

between 1 and 2
↓

$$\text{Since } 2n + 1 = n + (n + 1) \quad \therefore \lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$$

$$A \subseteq B \iff \underline{P(A) \subseteq P(B)}$$

($\rightarrow$) Assume $A \subseteq B$. *

Let $X \in P(A)$.

$\therefore X \subseteq A$ (**)

(*), (**)

$\therefore X \subseteq B$

$\therefore X \in P(B)$

$\therefore P(A) \subseteq P(B)$.

($\leftarrow$) Assume $P(A) \subseteq P(B)$

Since $A \subseteq A$

$\therefore A \in P(A)$

But $P(A) \subseteq P(B)$

$\uparrow$

$\therefore A \in P(B)$

$\therefore A \subseteq B$

$$X \in P(A) \equiv X \subseteq A$$

$$A \subseteq B \iff \underline{P(A) \subseteq P(B)}$$

($\rightarrow$) Assume $A \subseteq B$. *

Let $X \in P(A)$.

$\therefore X \subseteq A$ (**)

(*), (**)

$\therefore X \subseteq B$

$\therefore X \in P(B)$

$\therefore P(A) \subseteq P(B)$.

($\leftarrow$) Assume $P(A) \subseteq P(B)$

$$X \in P(A) \iff X \subseteq A$$

Example 2.3.4. Determine whether each function below is only injective, only surjective, or bijective.

(a) $\begin{cases} f : \mathbb{R} \rightarrow \mathbb{R} \\ f(x) = x^2 \end{cases}$

a)

Not surjective because the range of f is $\{y \in \mathbb{R} : y \geq 0\}$ which is not $\mathbb{R}$. In other words, some elements in $\mathbb{R}$ (such as -1) have no pre-image: $\nexists x \in \mathbb{R} : x^2 = -1$

Not injective, e.g. 4 has two pre-images: -2 and 2 . That is $f(-2) = f(2)$ but $-2 \neq 2$.

b) It's surjective, because $\forall b \in \mathbb{R}^+ : g(\sqrt{b}) = b$

It's injective, because $\forall a_1, a_2 \in \mathbb{R}^+ : \text{if } g(a_1) = g(a_2)$ then $a_1^2 = a_2^2 \therefore a_1 = a_2$ or $a_1 = -a_2$, but since $a_1, a_2 > 0$, we can't have $a_1 = -a_2 \therefore a_1 = a_2$
 $\therefore g$ is bijective.

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c) To show h is surjective, we need to show $\forall b \in \mathbb{R} \setminus \{1/2\} \quad \exists a \in \mathbb{R} \setminus \{3/2\}$ so that $h(a) = b$

This is true because by choosing

$$a = \frac{-3b}{1-2b}, \text{ we get } h(a) = b.$$

$$\frac{a}{2a-3} = b$$

(Solve for a in terms of b)

To show h is injective: